

SYMMETRIES, NEWTONOID VECTOR FIELDS AND CONSERVATION LAWS IN THE LAGRANGIAN k -SYMPLECTIC FORMALISM

LUCÍA BUA, IOAN BUCATARU, AND MODESTO SALGADO

ABSTRACT. In this paper we study symmetries, Newtonoid vector fields, conservation laws, Noether's Theorem and its converse, in the framework of the k -symplectic formalism, using the Frölicher-Nijenhuis formalism on the space of k^1 -velocities of the configuration manifold.

For the case $k = 1$, it is well known that Cartan symmetries induce and are induced by constants of motions, and these results are known as Noether's Theorem and its converse. For the case $k > 1$, we provide a new proof for Noether's Theorem, which shows that, in the k -symplectic formalism, each Cartan symmetry induces a conservation law. We prove that, under some assumptions, the converse of Noether's Theorem is also true and we provide examples when this is not the case. We also study the relations between dynamical symmetries, Newtonoid vector fields, Cartan symmetries and conservation laws, showing when one of them will imply the others. We use several examples of partial differential equations to illustrate when these concepts are related and when they are not.

1. INTRODUCTION

The k -symplectic formalism [39] is the generalization to field theories of the standard symplectic formalism in Mechanics [1, 2], which is the geometric framework for describing autonomous dynamical systems. A natural extension of this formalism is the so-called k -cosymplectic formalism, [33, 34], which is a generalization to field theories of the cosymplectic formalism describing non-autonomous mechanical systems. One of the advantages of using these formalisms is that only the tangent and cotangent bundles of the configuration manifold are required to develop them. Others papers related with the k -symplectic and k -cosymplectic formalism are [20, 27, 28, 37, 40, 46, 47].

The polysymplectic formalism developed by Giachetta, Mangiarotti and Sardanashvily in [14], which is based on a vector-valued form defined on some associated fiber bundle, is a different description of classical field theories of first order than the k -symplectic formalism. See also [21], for other considerations regarding this aspect. The soldering form on linear frames bundle is a polysymplectic form, and its study and applications to field theory, constitute the n -symplectic geometry developed by Norris in [41, 42, 43, 44].

Alternatively, one can derive the field equations by use of the so-called multisymplectic formalism, which was developed by Tulczyjew's school, [22, 23, 24, 48, 49], and independently by García and Pérez-Rendón [12, 13] and Goldschmidt and Sternberg [15]. This approach was revised by Gotay et al. [16, 17, 18, 19] and more recently by Cantrijn et al. [8, 9]. The relationship of the k -symplectic formalism with the multisymplectic formalism is studied in [46].

The aim of this paper is to study Noether's Theorem for first-order classical field theories, using the Lagrangian k -symplectic formalism. This study was initialized in [47] where large part of the discussion is a generalization of the results obtained for non-autonomous mechanical systems. See,

Date: March 3, 2013.

2000 Mathematics Subject Classification. 70S05, 70S10, 53D05.

Key words and phrases. Symmetries, Conservation laws, Noether's theorem, Lagrangian field theories, k -symplectic manifolds.

in particular [29] and references quoted therein. We introduce the set of Newtonoid vector fields and prove that any Cartan symmetry is a Newtonoid vector field. Furthermore, we show that under some assumptions, Newtonoid vector fields are Cartan symmetries and they induce conservation laws. This result extends the work developed by Marmo and Mukunda in [36]. The study of symmetries in field theory, using various geometric frameworks, has been done in [5, 11, 19, 30, 37].

The structure of the paper is as follows. In Section 2 we review the k -symplectic Lagrangian formalism, and hence the field theoretic state space of velocities is introduced in Section 2.1 as the Whitney sum $T_k^1 Q$ of k -copies of the tangent bundle TQ of a manifold Q . This manifold has a canonical k -tangent structure defined by k tensor fields (J^1, \dots, J^k) of type $(1, 1)$, see [31, 32]. In the case $k = 1$, J^1 is the canonical tangent structure of the tangent bundle TQ . The canonical k -tangent structure of $T_k^1 Q$ is used to construct the Poincaré-Cartan forms.

A particular type of second order partial differential equations, which we call SOPDE, are introduced in Section 2.2. They are a generalization of SODE's (semisprays) found in Geometric Mechanics. The Lagrangian formalism is developed in Section 2.3.

In Section 3 we discuss symmetries and conservation laws for Lagrangian functions on $T_k^1 Q$. We prove Noether's Theorem 3.9, which shows that each Cartan symmetry induces a conservation law. We provide in Proposition 3.11 some conditions under which the converse of Noether's Theorem is true. Noether's Theorem 3.9 was proved previously in [47] using local coordinates. Here we present a direct global proof using the Frölicher-Nijenhuis formalism. For a modern description of the Frölicher-Nijenhuis formalism see [25, §8]. In Section 3.2 we introduce the set of Newtonoid vector fields in the framework of k -symplectic formalism, extending the work of Marmo and Mukunda [36] for the case $k = 1$. In Proposition 3.8 we prove that Cartan symmetries are always Newtonoid vector fields. In Theorem 3.13 we show that, under some assumptions, Newtonoid vector fields are Cartan symmetries and hence they provide conservation laws.

2. REVIEW OF LAGRANGIAN k -SYMPLECTIC FORMALISM

In this section we briefly recall the Lagrangian k -symplectic formalism. We refer the reader to [39, 47] for more details about this formalism. We present first the geometric framework for this formalism, which is given by the tangent bundle of k^1 -velocities $T_k^1 Q$ of the configuration manifold Q , together with the canonical structures. For a Lagrangian on $T_k^1 Q$, the geometric informations we need for the Lagrangian k -symplectic formalism are encoded in the Poincaré-Cartan forms. We discuss further systems of second order partial differential equations (SOPDE) as well as their relations with Euler-Lagrange equations.

2.1. Geometric framework.

The tangent bundle of k^1 -velocities of a manifold. Canonical structures. In this work we consider Q a real, n -dimensional and C^∞ -smooth manifold. Throughout the paper, we assume that all objects are C^∞ -smooth where defined. Consider (TQ, τ, Q) the tangent bundle of the manifold Q . We denote by $C^\infty(Q)$ the ring of smooth functions on Q , and by $\mathfrak{X}(Q)$ the $C^\infty(Q)$ -module of vector fields on Q .

Let us denote by $T_k^1 Q$ the Whitney sum $TQ \oplus \dots \oplus TQ$ of k copies of TQ , with projection $\tau_Q: T_k^1 Q \rightarrow Q$. $T_k^1 Q$ can be identified with the manifold $J_0^1(\mathbb{R}^k, Q)$ of the k^1 -velocities of Q ; that is, 1-jets of maps $\sigma: \mathbb{R}^k \rightarrow Q$, with source at $0 \in \mathbb{R}^k$. For this reason the manifold $T_k^1 Q$ is called *the tangent bundle of k^1 -velocities of Q* , see [38]. If (q^i) are local coordinates on $U \subseteq Q$, then the induced local coordinates in $T_k^1 U = \tau_Q^{-1}(U)$ are denoted by (q^i, v_α^i) , $1 \leq i \leq n$, $1 \leq \alpha \leq k$. Throughout this work we implicitly assume summation over repeated covariant and contravariant latin indices $i, j, l, \dots \in \{1, \dots, n\}$, as well as summation over repeated greek indices $\alpha, \beta, \dots \in \{1, \dots, k\}$.

The *canonical k -tangent structure* on $T_k^1 Q$, see [39], is the family $J = (J^1, \dots, J^k)$ of k tensor fields of type $(1, 1)$, which are locally given by

$$(2.1) \quad J^\alpha = \frac{\partial}{\partial v_\alpha^i} \otimes dq^i, \quad \alpha \in \{1, \dots, k\}.$$

In the case $k = 1$, J^1 is the well-known canonical tangent structure of the tangent bundle.

The *Liouville vector field*, $\mathbb{C} \in \mathfrak{X}(T_k^1 Q)$, is the infinitesimal generator of the following flow

$$\psi: \mathbb{R} \times T_k^1 Q \longrightarrow T_k^1 Q, \quad \psi(s, (q, v_1, \dots, v_k)) = (q, e^s v_1, \dots, e^s v_k).$$

In local coordinates, the Liouville vector field has the form

$$(2.2) \quad \mathbb{C} = v_\alpha^i \frac{\partial}{\partial v_\alpha^i}.$$

The *vertical distribution* is the kn -dimensional distribution on $T_k^1 Q$ given by $V: u \in T_k^1 Q \rightarrow V(u) = \text{Ker } d_u \tau_Q = \text{Ker } J_u \subset T_u T_k^1 Q$. The vertical distribution V splits into k subdistributions $V^\alpha(u) = \text{Im } J^\alpha(u)$, $\alpha \in \{1, \dots, k\}$. Each of these vertical subdistributions are n -dimensional and integrable since $V^\alpha(u) = \text{span} \{ \partial/\partial v_\alpha^i, 1 \leq i \leq n \}$.

Poincaré-Cartan forms on $T_k^1 Q$. The Lagrangian k -symplectic formalism for a Lagrangian function L on $T_k^1 Q$ can be developed from the corresponding Poincaré-Cartan forms.

Definition 2.1. A *Lagrangian* is a smooth function L on $T_k^1 Q$. A Lagrangian $L \in C^\infty(T_k^1 Q)$ is called *regular* if the Hessian matrix of L with respect to the fibre coordinates,

$$(2.3) \quad g_{ij}^{\alpha\beta}(q, v) = \frac{\partial^2 L}{\partial v_\alpha^i \partial v_\beta^j}(q, v)$$

has maximal rank kn on $T_k^1 Q$.

For a Lagrangian L , the *energy function* is $E_L = \mathbb{C}(L) - L \in C^\infty(T_k^1 Q)$, with local expression

$$(2.4) \quad E_L = v_\alpha^i \frac{\partial L}{\partial v_\alpha^i} - L.$$

For each Lagrangian $L \in C^\infty(T_k^1 Q)$ we consider the family of *Poincaré-Cartan 1-forms* $\theta_L^\alpha = dJ^\alpha L = dL \circ J^\alpha$, as well as the family of Poincaré-Cartan 2-forms on $T_k^1 Q$, $\omega_L^\alpha = -d\theta_L^\alpha$.

In induced local coordinates on $T_k^1 Q$, the Poincaré-Cartan forms are given by

$$(2.5) \quad \begin{aligned} \theta_L^\alpha &= \frac{\partial L}{\partial v_\alpha^i} dq^i, \\ \omega_L^\alpha &= \frac{1}{2} \left(\frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} - \frac{\partial^2 L}{\partial q^i \partial v_\alpha^j} \right) dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v_\beta^j \partial v_\alpha^i} dq^i \wedge dv_\beta^j. \end{aligned}$$

We recall now the definition of a k -symplectic structure, see [3, 4].

Definition 2.2. A *k -symplectic* structure on a $k+kn$ -dimensional manifold M is given by a family of k , closed 2-forms $(\omega^1, \dots, \omega^k)$ and an integrable kn -dimensional distribution V on M such that

$$i) \cap_{\alpha=1}^k \text{Ker}(\omega^\alpha) = \{0\}, \quad ii) \omega_{|_{V \times V}}^\alpha = 0, \quad \alpha \in \{1, \dots, k\}.$$

Using formulae (2.3) and (2.5) one obtains that a Lagrangian L is regular if and only if the Poincaré-Cartan 2-forms and the vertical distribution, $(\omega_L^1, \dots, \omega_L^k, V = \text{ker } \tau_Q)$, define a k -symplectic structure on $T_k^1 Q$, see [39].

Complete lifts of vector fields. The lifting process of some geometric structures from a base manifold to the total space of some fibre bundle has proven its usefulness for studying the corresponding geometric structures [38, 50]. For example, the complete lift of a system of second order ordinary differential equations contains informations about its symmetries and first order variations, [7].

Definition 2.3. Let $\phi: Q \rightarrow Q$ be a differentiable map, then the *first order prolongation* of ϕ to $T_k^1 Q$ is the map $T_k^1 \phi: T_k^1 Q \rightarrow T_k^1 Q$, defined by $T_k^1 \phi(j_0^1 \sigma) = j_0^1(\phi \circ \sigma)$. This means that for $(q, v_1, \dots, v_k) \in T_q Q$, $q \in Q$, we have

$$T_k^1 \phi(q, v_1, \dots, v_k) = (d_q \phi(v_1), \dots, d_q \phi(v_k)).$$

If Z is a vector field on Q , with local 1-parametric group of transformations $h_s: Q \rightarrow Q$, then the local 1-parametric group of transformations $T_k^1 h_s: T_k^1 Q \rightarrow T_k^1 Q$ generates a vector field Z^C on $T_k^1 Q$, which is called the *complete lift* of Z to $T_k^1 Q$. If locally $Z = Z^i \partial / \partial q^i$, then the complete lift is given by

$$Z^C = Z^i \frac{\partial}{\partial q^i} + v_\alpha^i \frac{\partial Z^j}{\partial q^i} \frac{\partial}{\partial v_\alpha^j}.$$

If we consider also the vertical lifts $Z^{V_\alpha} = J^\alpha Z^C$, then the following properties are well known, see [38],

$$[X^C, Y^C] = [X, Y]^C, \quad [X^C, Y^{V_\alpha}] = [X, Y]^{V_\alpha}, \quad [X^{V_\alpha}, X^{V_\beta}] = 0.$$

These formulae extend the well known properties of Lie brackets for vertical and complete lifts of vector fields to TQ , [50].

2.2. Systems of first and second-order partial differential equations. A vector field on a manifold M defines a system of first-order ordinary differential equations. Accordingly, a k -vector field on M , for some $k > 1$, defines a system of first-order partial differential equations. Furthermore, some special k -vector field on the manifold $M = T_k^1 Q$ defines a system of second-order partial differential equations.

First-order partial differential equations on a manifold. In this subsection, we briefly show how k -vector fields determine systems of first-order partial differential equations.

Definition 2.4. A k -vector field on an arbitrary manifold M is a section $X: M \rightarrow T_k^1 M$ of the canonical projection $\tau_M: T_k^1 M \rightarrow M$.

Since $T_k^1 M$ is the Whitney sum $TM \oplus \dots \oplus TM$ of k copies of TM , we deduce that a k -vector field X defines a family of k vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$ by projecting X onto every factor; that is, $X_\alpha = \tau_\alpha \circ X$, where $\tau_\alpha: T_k^1 M \rightarrow TM$ is the canonical projection on the α^{th} -copy TM of $T_k^1 M$.

Definition 2.5. An *integral section* of the k -vector field $X = (X_1, \dots, X_k)$, passing through a point $x \in M$, is a map $\psi: U \subset \mathbb{R}^k \rightarrow M$, defined on some open neighborhood U of $0 \in \mathbb{R}^k$, such that

$$\psi(0) = x, \quad d_t \psi \left(\frac{\partial}{\partial t^\alpha} \Big|_t \right) = X_\alpha(\psi(t)) \in T_{\psi(t)} M, \quad \text{for every } t \in U, \quad 1 \leq \alpha \leq k.$$

Equivalently, ψ satisfies $X \circ \psi = \psi^{(1)}$, where $\psi^{(1)}: U \subset \mathbb{R}^k \rightarrow T_k^1 M$ is the first-order prolongation of ψ to $T_k^1 M$ defined by

$$(2.6) \quad \begin{aligned} \psi^{(1)}: U \subset \mathbb{R}^k &\rightarrow T_k^1 M \\ t &\rightarrow \psi^{(1)}(t) = \left(d_t \psi \left(\frac{\partial}{\partial t^1} \Big|_t \right), \dots, d_t \psi \left(\frac{\partial}{\partial t^k} \Big|_t \right) \right). \end{aligned}$$

In local coordinates, if $\psi(t) = (\psi^i(t))$, then we have

$$(2.7) \quad \psi^{(1)}(t) = \left(\psi^i(t), \frac{\partial \psi^i}{\partial t^\alpha}(t) \right).$$

A k -vector field $X = (X_1, \dots, X_k)$ on M is *integrable* if there is an integral section passing through every point of M .

Consider $X = (X_1, \dots, X_k)$ a k -vector field, where $X_\alpha = X_\alpha^i \partial / \partial x^i$ in a coordinate system (U, x^i) on M . The k -vector field X induces a system of first-order partial differential equations on M , which is given by

$$X_\alpha^i(x^j(t)) = \frac{\partial x^i}{\partial t^\alpha} \Big|_t, \quad \alpha \in \{1, \dots, k\}, \quad i \in \{1, \dots, \dim M\}.$$

From Definition 2.5 we deduce that ψ is an integral section of $X = (X_1, \dots, X_k)$ if ψ is a solution to the above system of first-order partial differential equations, which means that it satisfies

$$X_\alpha^i(\psi(t)) = \frac{\partial \psi^i}{\partial t^\alpha} \Big|_t, \quad \alpha \in \{1, \dots, k\}, \quad i \in \{1, \dots, \dim M\}.$$

Systems of second-order partial differential equations. In this part we characterize those integrable k -vector fields on $M = T_k^1 Q$ that have as integral sections first order prolongations $\phi^{(1)}$ of maps $\phi : U \subset \mathbb{R}^k \rightarrow Q$. Such k -vector fields define integrable systems of second-order partial differential equations on the base manifold Q .

As we recalled, a k -vector field in $T_k^1 Q$ is a section $\xi : T_k^1 Q \rightarrow T_k^1(T_k^1 Q)$ of the canonical projection $\tau_{T_k^1 Q} : T_k^1(T_k^1 Q) \rightarrow T_k^1 Q$. We note that there are systems of partial differential equations that are not induced by k -vector fields. However, in this work we are interested only in those systems of PDE that are induced by such k -vector fields. In view of these considerations, we consider the following definition.

Definition 2.6. A system of *second-order partial differential equations* (SOPDE) on Q is a k -vector field $\xi = (\xi_1, \dots, \xi_k)$ on $T_k^1 Q$, which is a section of the projection $T_k^1 \tau_Q : T_k^1(T_k^1 Q) \rightarrow T_k^1 Q$, namely $T_k^1 \tau_Q \circ \xi = \text{Id}_{T_k^1 Q}$, and this is equivalent to

$$(2.8) \quad d\tau_Q \circ \xi_\alpha = \tau_\alpha : T_k^1 Q \rightarrow TQ, \quad \alpha \in \{1, \dots, k\}.$$

Equivalently, above equations can be written as follows

$$d_{(q,v)} \tau_Q(\xi_\alpha(q, v)) = (q, v_\alpha), \quad \text{for } (q, v_1, \dots, v_k) \in T_k^1 Q, \quad \alpha \in \{1, \dots, k\}.$$

In the case $k = 1$, Definition 2.6 reduces to the definition of a system of second-order ordinary differential equations (SODE).

Locally, a SOPDE $\xi = (\xi_1, \dots, \xi_k)$ is given by

$$(2.9) \quad \xi_\alpha = v_\alpha^i \frac{\partial}{\partial q^i} + \xi_{\alpha\beta}^i \frac{\partial}{\partial v_\beta^i}, \quad \alpha \in \{1, \dots, k\},$$

where $\xi_{\alpha\beta}^i$ are smooth functions defined on domains of induced charts on $T_k^1 Q$.

All these considerations allow us to reformulate the definition for a SOPDE, using the k -tangent structure and the Liouville vector field \mathbb{C} (see formulae (2.1) and (2.2)), as follows.

Proposition 2.7. A k -vector field $\xi = (\xi_1, \dots, \xi_k)$ on $T_k^1 Q$ is a SOPDE if and only if $J^\alpha(\xi_\alpha) = \mathbb{C}$.

If $\psi : U \subset \mathbb{R}^k \rightarrow T_k^1 Q$, locally given by $\psi(t) = (\psi^i(t), \psi_\alpha^i(t))$, is an integral section of a SOPDE $\xi = (\xi_1, \dots, \xi_k)$ then from Definition 2.5 and formula (2.9) it follows

$$(2.10) \quad \frac{\partial \psi^i}{\partial t^\alpha} \Big|_t = \psi_\alpha^i(t), \quad \frac{\partial \psi_\alpha^i}{\partial t^\beta} \Big|_t = \xi_{\alpha\beta}^i(\psi(t)).$$

Using formulae (2.7) and (2.10) we obtain the following characterization for the integral maps of a SOPDE.

Proposition 2.8. *Let $\xi = (\xi_1, \dots, \xi_k)$ be an integrable SOPDE. If ψ is an integral section of ξ , then $\psi = \phi^{(1)}$, where $\phi^{(1)}$ is the first-order prolongation of the map*

$$\phi = \tau_Q \circ \psi : U \subset \mathbb{R}^k \xrightarrow{\psi} T_k^1 Q \xrightarrow{\tau_Q} Q,$$

and ϕ is a solution to the system of second-order partial differential equations

$$(2.11) \quad \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta}(t) = \xi_{\alpha\beta}^i \left(\phi^j(t), \frac{\partial \phi^j}{\partial t^\gamma}(t) \right).$$

Conversely, if $\phi : U \subset \mathbb{R}^k \rightarrow Q$ is any map satisfying the system (2.11), then $\phi^{(1)}$ is an integral section of $\xi = (\xi_1, \dots, \xi_k)$.

Definition 2.9. If $\phi^{(1)}$ is an integral section of a SOPDE $\xi = (\xi_1, \dots, \xi_k)$, then map ϕ will be called a *solution* to ξ .

From equations (2.11) we deduce that if ξ is an integrable SOPDE then necessarily we have the symmetry $\xi_{\alpha\beta}^i = \xi_{\beta\alpha}^i$ for all $\alpha, \beta = 1, \dots, k$. Therefore, for a SOPDE ξ , locally given by formula (2.9), we require the following integrability conditions [26]:

$$(2.12) \quad \xi_{\alpha\beta}^i = \xi_{\beta\alpha}^i, \quad \xi_\alpha(\xi_{\beta\gamma}^i) = \xi_\beta(\xi_{\alpha\gamma}^i), \forall \alpha, \beta, \gamma \in \{1, \dots, k\}.$$

Integrability conditions (2.12) are equivalent to the fact that $[\xi_\alpha, \xi_\beta] = 0$, $\forall \alpha, \beta \in \{1, \dots, k\}$. The integrability conditions (2.12) have been also proved in [35]. Due to the first symmetry condition (2.12) we have that the system (2.11) is a system of $nk(k+1)/2$ second-order partial differential equations.

2.3. Euler-Lagrange equations. An important class of SOPDE on a manifold Q contains those whose solutions are among the solutions of the Euler-Lagrange equations for some Lagrangian function on $T_k^1 Q$. In Proposition 2.11 we characterize this class, while in Proposition 2.12 we discuss the relation between the solutions of a SOPDE in this class and the solutions of the corresponding Euler-Lagrange equations.

The variational problem for a Lagrangian L on $T_k^1 Q$ leads to the following system of Euler-Lagrange equations

$$(2.13) \quad \frac{\partial}{\partial t^\alpha} \left(\frac{\partial L}{\partial v_\alpha^i} \right) - \frac{\partial L}{\partial q^i} = 0.$$

Euler-Lagrange equations (2.13) can be written as

$$(2.14) \quad g_{ij}^{\alpha\beta} \frac{\partial^2 q^j}{\partial t^\alpha \partial t^\beta} + \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} v_\alpha^j - \frac{\partial L}{\partial q^i} = 0,$$

which represents a system of n second-order partial differential equations on Q .

Denote by $\mathfrak{X}_L^k(T_k^1 Q)$ the set of k -vector fields $\xi = (\xi_1, \dots, \xi_k)$ on $T_k^1 Q$, which are solutions to the equation

$$(2.15) \quad i_{\xi_\alpha} \omega_L^\alpha = dE_L.$$

If each ξ_α is locally given by

$$(2.16) \quad \xi_\alpha = \xi_\alpha^i \frac{\partial}{\partial q^i} + \xi_{\alpha\beta}^i \frac{\partial}{\partial v_\beta^i}, \quad \alpha \in \{1, \dots, k\},$$

then (ξ_1, \dots, ξ_k) is a solution to (2.15) if and only if the functions ξ_α^i and $\xi_{\alpha\beta}^i$ satisfy the following system of equations

$$(2.17) \quad \begin{aligned} \left(\frac{\partial^2 L}{\partial q^i \partial v_\alpha^j} - \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} \right) \xi_\alpha^j - \frac{\partial^2 L}{\partial v_\alpha^i \partial v_\beta^j} \xi_{\alpha\beta}^j &= v_\alpha^j \frac{\partial^2 L}{\partial q^i \partial v_\alpha^j} - \frac{\partial L}{\partial q^i} \\ \frac{\partial^2 L}{\partial v_\beta^j \partial v_\alpha^i} \xi_\alpha^i &= \frac{\partial^2 L}{\partial v_\beta^j \partial v_\alpha^i} v_\alpha^i. \end{aligned}$$

If L is a regular Lagrangian, the above equations are equivalent to the following equations

$$(2.18) \quad \frac{\partial^2 L}{\partial v_\alpha^i \partial v_\beta^j} \xi_{\alpha\beta}^j + \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} v_\alpha^j - \frac{\partial L}{\partial q^i} = 0, \quad \xi_\alpha^i = v_\alpha^i.$$

Using equations (2.17) we deduce the following lemma.

Lemma 2.10. *Consider $L \in C^\infty(T_k^1 Q)$ a Lagrangian.*

- 1) *If L is regular, then any k -vector field $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$ is a SOPDE, it is locally given by formula (2.9) and satisfies equations (2.18).*
- 2) *If $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$ and ξ is a SOPDE, then it is locally given by formula (2.9) and satisfies equations (2.18).*

Next proposition characterizes the set of SOPDEs that are in $\mathfrak{X}_L^k(T_k^1 Q)$.

Proposition 2.11. *Let $L \in C^\infty(T_k^1 Q)$ be a Lagrangian and let $\xi = (\xi_1, \dots, \xi_k)$ be a SOPDE on $T_k^1 Q$. Then $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$ if and only if it satisfies the following condition:*

$$(2.19) \quad \mathcal{L}_{\xi_\alpha} \theta_L^\alpha = dL, \quad \text{or locally} \quad \xi_\alpha \left(\frac{\partial L}{\partial v_\alpha^i} \right) = \frac{\partial L}{\partial q^i}.$$

Proof. We will start by proving that the first equation in (2.19) is equivalent to equation (2.15). Since ξ is a SOPDE we have that $J^\alpha \xi_\alpha = \mathbb{C}$ and hence it follows that $i_{\xi_\alpha} \theta_L^\alpha = \theta_L^\alpha(\xi_\alpha) = (dL \circ J^\alpha)(\xi_\alpha) = \mathbb{C}L$. Using the fact that $\omega_L^\alpha = -d\theta_L^\alpha$ we obtain

$$\mathcal{L}_{\xi_\alpha} \theta_L^\alpha = di_{\xi_\alpha} \theta_L^\alpha + i_{\xi_\alpha} d\theta_L^\alpha = d(\mathbb{C}L) - i_{\xi_\alpha} \omega_L^\alpha = dL + (dE_L - i_{\xi_\alpha} \omega_L^\alpha).$$

It follows that the first equation in (2.19) is equivalent to equation (2.15).

Since ξ is a SOPDE it follows that $\xi_\alpha(q^i) = v_\alpha^i$. Therefore, we have

$$\begin{aligned} \mathcal{L}_{\xi_\alpha} \theta_L^\alpha - dL &= \mathcal{L}_{\xi_\alpha} \left(\frac{\partial L}{\partial v_\alpha^i} dq^i \right) - dL = \xi_\alpha \left(\frac{\partial L}{\partial v_\alpha^i} \right) dq^i + \left(\frac{\partial L}{\partial v_\alpha^i} \right) dv_\alpha^i - dL \\ &= \left[\xi_\alpha \left(\frac{\partial L}{\partial v_\alpha^i} \right) - \frac{\partial L}{\partial q^i} \right] dq^i, \end{aligned}$$

and hence the two equations in (2.19) are equivalent. \square

We will discuss now the relation between solutions of the Euler-Lagrange equations (2.13) or (2.14) and integral sections of k -vector fields in $\mathfrak{X}_L^k(T_k^1 Q)$.

Proposition 2.12. *Consider a Lagrangian L on $T_k^1 Q$ and a k -vector field $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$.*

- 1) *If ξ is a SOPDE, then a map $\phi : U \subset \mathbb{R}^k \rightarrow Q$ is a solution to the Euler-Lagrange equations (2.13) if and only if*

$$(2.20) \quad g_{ij}^{\alpha\beta} \circ \phi^{(1)} \left(\xi_{\alpha\beta}^j \circ \phi^{(1)} - \frac{\partial^2 \phi^j}{\partial t^\alpha \partial t^\beta} \right) = 0.$$

- 2) *If the Lagrangian L is regular, then ξ is a SOPDE, and if $\phi : U \subset \mathbb{R}^k \rightarrow Q$ is a solution to ξ , then ϕ is a solution to the Euler-Lagrange equations (2.13).*

- 3) If ξ is integrable, and $\phi^{(1)} : U \subset \mathbb{R}^k \rightarrow T_k^1 Q$ is an integral section, then $\phi : U \subset \mathbb{R}^k \rightarrow Q$ is a solution to the Euler-Lagrange equations (2.13).

Proof. 1) Consider a map $\phi : U \subset \mathbb{R}^k \rightarrow Q$. If ϕ is a solution to the Euler-Lagrange equations (2.14), then we have

$$(2.21) \quad g_{ij}^{\alpha\beta} \circ \phi^{(1)} \frac{\partial^2 \phi^j}{\partial t^\alpha \partial t^\beta} + \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} \circ \phi^{(1)} \frac{\partial \phi^j}{\partial t^\alpha} - \frac{\partial L}{\partial q^i} \circ \phi^{(1)} = 0.$$

If the k -vector field ξ is a SOPDE, then $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$ if and only if it satisfies the equations

$$(2.22) \quad \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} v_\alpha^j + \frac{\partial^2 L}{\partial v_\alpha^i \partial v_\beta^j} \xi_{\alpha\beta}^j = \frac{\partial L}{\partial q^i}.$$

If we restrict equation (2.22) to the image of $\phi^{(1)}$ we obtain

$$(2.23) \quad g_{ij}^{\alpha\beta} \circ \phi^{(1)} \xi_{\alpha\beta}^j \circ \phi^{(1)} + \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} \circ \phi^{(1)} \frac{\partial \phi^j}{\partial t^\alpha} - \frac{\partial L}{\partial q^i} \circ \phi^{(1)} = 0.$$

Using equations (2.23) it follows that ϕ satisfies (2.20) if and only if it satisfies (2.21) that are equivalent to Euler-Lagrange equations (2.14).

2) If $\phi : U \subset \mathbb{R}^k \rightarrow Q$ is a solution to ξ then it satisfies equations (2.11). Therefore, equations (2.20) are automatically satisfied and hence ϕ is a solution of the Euler-Lagrange equations (2.14).

3) Since $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$ it follows that ξ satisfies first equation (2.17). If we restrict this equation to $\phi^{(1)} : U \subset \mathbb{R}^k \rightarrow T_k^1 Q$, which is an integral map of ξ , we obtain that ϕ satisfies the Euler-Lagrange equations (2.14). \square

Remark 2.13. The results of Lemma 2.10 and results 2) and 3) of Proposition 2.12 are the fundamentals of Lagrangian k -symplectic formalism and equation (2.15) can be seen as a geometric version of the Euler-Lagrange field equations.

Remark 2.14. Formula (2.20) does not require any relationship between the k -vector field $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$ and the solution ϕ to the Euler-Lagrange equations (2.13). In other words, we might have ϕ a solution to the Euler-Lagrange equations (2.13) which may not be a solution for any $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$

Example 2.15. In this example we consider the theory of a vibrating string. Coordinates (t^1, t^2) are interpreted as the time and the distance along the string, respectively. If $\phi : (t^1, t^2) \in \mathbb{R}^2 \rightarrow \phi(t^1, t^2) \in \mathbb{R}$ denotes the displacement of each point of the string as function of the time t^1 and the position t^2 , the motion equation is

$$(2.24) \quad \sigma \frac{\partial^2 \phi}{\partial (t^1)^2} - \tau \frac{\partial^2 \phi}{\partial (t^2)^2} = 0,$$

where σ and τ are certain constants of the mechanical system.

Equation (2.24) is the Euler-Lagrange equation for the regular Lagrangian

$$(2.25) \quad L : T_2^1 \mathbb{R} \rightarrow \mathbb{R}, \quad L(q, v_1, v_2) = \frac{1}{2}(\sigma v_1^2 - \tau v_2^2).$$

From formulae (2.5) and (2.25) we deduce that

$$(2.26) \quad \omega_L^1 = \sigma dq \wedge dv_1, \quad \omega_L^2 = -\tau dq \wedge dv_2, \quad dE_L = \sigma v_1 dv_1 - \tau v_2 dv_2.$$

Therefore, a SOPDE $(\xi_1, \xi_2) \in \mathfrak{X}(T_2^1 \mathbb{R})$

$$\xi_1 = v_1 \frac{\partial}{\partial q} + \xi_{11} \frac{\partial}{\partial v_1} + \xi_{12} \frac{\partial}{\partial v_2}, \quad \xi_2 = v_2 \frac{\partial}{\partial q} + \xi_{12} \frac{\partial}{\partial v_1} + \xi_{22} \frac{\partial}{\partial v_2},$$

is a solution to equation (2.15) if and only if it satisfies

$$(2.27) \quad \sigma\xi_{11} - \tau\xi_{22} = 0.$$

The integrability conditions (2.12) are in this case

$$(2.28) \quad \frac{\partial\xi_{11}}{\partial v_1} = \frac{\partial\xi_{12}}{\partial v_2}, \quad \sigma\frac{\partial\xi_{11}}{\partial v_2} = \tau\frac{\partial\xi_{12}}{\partial v_1}.$$

An example of an integrable SOPDE, which is a solution to (2.27) is given by

$$\begin{aligned} \xi_1 &= v_1 \frac{\partial}{\partial q} + \tau (\sigma(v_1)^2 + \tau(v_2)^2) \frac{\partial}{\partial v_1} + 2\sigma\tau v_1 v_2 \frac{\partial}{\partial v_2}, \\ \xi_2 &= v_2 \frac{\partial}{\partial q} + 2\sigma\tau v_1 v_2 \frac{\partial}{\partial v_1} + \sigma (\sigma(v_1)^2 + \tau(v_2)^2) \frac{\partial}{\partial v_2}. \end{aligned}$$

Thus any solution ϕ of the integrable SOPDE (ξ_1, ξ_2) in the formulae above is a solution of the vibrating string equation (2.24).

3. NOETHER'S THEOREM

In this section we discuss symmetries and conservation laws for Lagrangian functions on $T_k^1 Q$. We introduce the Newtonoid vector fields in this framework, extending the work of Marmo and Mukunda [36] for the case $k = 1$. We provide a new proof for Noether's Theorem 3.9 as well as some conditions under which its converse is true. Noether's Theorem 3.9 was proved previously in [47] using local coordinates. Here we present a direct global proof using the Frölicher-Nijenhuis formalism.

3.1. Conservation laws and Cartan symmetries. For a regular Lagrangian on TQ , the corresponding Euler-Lagrange equations are equivalent to a SODE. This implies that its dynamical symmetries are equivalent to Cartan symmetries, which (locally) determine and are determined by constants of motions [10, §13.8]. For $k > 1$ and a Lagrangian L on $T_k^1 Q$ none of the above equivalences are true anymore in the very general context. However, some relations remain true. In this subsection we discuss such relations between Cartan symmetries and conservation laws.

Definition 3.1. A map $f = (f^1, \dots, f^k): T_k^1 Q \rightarrow \mathbb{R}^k$ is called a *conservation law* (or a *conserved quantity*) for the Euler-Lagrange equations (2.13) if the divergence of

$$f \circ \phi^{(1)} = (f^1 \circ \phi^{(1)}, \dots, f^k \circ \phi^{(1)}): U \subset \mathbb{R}^k \rightarrow \mathbb{R}^k$$

is zero, for every $\phi: U \subset \mathbb{R}^k \rightarrow M$ solution to the Euler-Lagrange equations (2.13), that is

$$(3.1) \quad 0 = \frac{\partial(f^\alpha \circ \phi^{(1)})}{\partial t^\alpha} \Big|_t = \frac{\partial f^\alpha}{\partial q^i} \Big|_{\phi^{(1)}(t)} \frac{\partial \phi^i}{\partial t^\alpha} \Big|_t + \frac{\partial f^\alpha}{\partial v_\beta^i} \Big|_{\phi^{(1)}(t)} \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta} \Big|_t.$$

Now, we present a simple example of conservation law.

Example 3.2. The following two functions $f^\alpha: T_2^1 \mathbb{R} \rightarrow \mathbb{R}$, $\alpha \in \{1, 2\}$, where

$$(3.2) \quad f^1(v_1, v_2) = -2\sigma v_1 v_2, \quad f^2(v_1, v_2) = \sigma(v_1)^2 + \tau(v_2)^2,$$

give a conservation law for the Euler-Lagrange equation (2.24). In fact, if ϕ is a solution to the Euler-Lagrange equations (2.24), using (3.2) we deduce that

$$\frac{\partial(f^1 \circ \phi^{(1)})}{\partial t^1} + \frac{\partial(f^2 \circ \phi^{(1)})}{\partial t^2} = 0.$$

Hence the functions (3.2) give a conservation law for the Lagrangian (2.25).

Lemma 3.3. Let $f = (f^1, \dots, f^k) : T_k^1 Q \rightarrow \mathbb{R}^k$ be a conservation law. Let $\xi = (\xi_1, \dots, \xi_k)$ be an integrable SOPDE in $\mathfrak{X}_L^k(T_k^1 Q)$, then

$$(3.3) \quad \xi_\alpha(f^\alpha) = 0.$$

Proof. Since ξ is integrable we know that for every point $x \in T_k^1 Q$ there exists an integral section $\phi^{(1)} : U \subset \mathbb{R}^k \rightarrow T_k^1 Q$ such that

- 1) ϕ is a solution to the Euler-Lagrange equations, because $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$,
- 2) ϕ satisfies

$$\phi^{(1)}(0) = x, \quad d_t \phi^{(1)} \left(\frac{\partial}{\partial t^\alpha} \Big|_t \right) = \xi_\alpha(\phi^{(1)}(t)) \in T_{\phi^{(1)}(t)}(T_k^1 Q),$$

for every $t \in U$ and for all $\alpha \in \{1, \dots, k\}$.

Condition 2) above means that

$$(3.4) \quad v_\alpha^i(\phi^{(1)}(t)) = \frac{\partial \phi^i}{\partial t^\alpha} \Big|_t, \quad \xi_{\alpha\beta}^i(\phi^{(1)}(t)) = \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta} \Big|_t.$$

Since $f = (f^1, \dots, f^k)$ is a conservation law, using formula (3.1) at $t = 0$, and using formulae (3.4), we have

$$\begin{aligned} 0 &= \frac{\partial(f^\alpha \circ \phi^{(1)})}{\partial t^\alpha} \Big|_0 = \frac{\partial f^\alpha}{\partial q^i} \Big|_{\phi^{(1)}(0)} \frac{\partial \phi^i}{\partial t^\alpha} \Big|_0 + \frac{\partial f^\alpha}{\partial v_\beta^i} \Big|_{\phi^{(1)}(0)} \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta} \Big|_0 \\ &= \frac{\partial f^\alpha}{\partial q^i} \Big|_x \frac{\partial \phi^i}{\partial t^\alpha} \Big|_0 + \frac{\partial f^\alpha}{\partial v_\beta^i} \Big|_x \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta} \Big|_0 = \frac{\partial f^\alpha}{\partial q^i} \Big|_x v_\alpha^i(x) + \frac{\partial f^\alpha}{\partial v_\beta^i} \Big|_x \xi_{\alpha\beta}^i(x) = \xi_\alpha(x) f^\alpha \end{aligned}$$

□

The converse of Lemma 3.3 may not be true, and the reason is that, as we can see from formula (2.20), we might have solutions ϕ of the Euler-Lagrange equations (2.14) that are not solutions to some $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$.

However, we will see in the following Lemma that, under some assumption on functions f^α , this converse is true.

Lemma 3.4. Let $L \in C^\infty(T_k^1 Q)$ be a Lagrangian and assume that there exists a vector field $X \in \mathfrak{X}(T_k^1 Q)$ such that

$$(3.5) \quad i_X \omega_L^\alpha = df^\alpha, \quad \forall \alpha \in \{1, \dots, k\},$$

for some functions $f^\alpha : T_k^1 Q \rightarrow \mathbb{R}$.

Then, f^α is a conservation law for the Euler-Lagrange equations (2.13) if and only if $\xi_\alpha(f^\alpha) = 0$, for all integrable SOPDE $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$.

Proof. The direct implication is given by Lemma 3.3. Note that for this implication we do not need the assumption on the existence of the vector field X that satisfies (3.5).

For the converse implication consider $X = X^i \partial / \partial q^i + X_\alpha^i \partial / \partial v_\alpha^i$ a vector field on $T_k^1 Q$ that satisfies (3.5). In view of the second formula (2.5) we can write equation (3.5) as follows

$$\left[\left(\frac{\partial^2 L}{\partial q^i \partial v_\alpha^j} - \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} \right) X^j - g_{ij}^{\alpha\beta} X_\beta^j \right] dq^i + g_{ij}^{\alpha\beta} X^i dv_\beta^j = \frac{\partial f^\alpha}{\partial q^i} dq^i + \frac{\partial f^\alpha}{\partial v_\beta^j} dv_\beta^j,$$

and necessarily we have

$$(3.6) \quad \frac{\partial f^\alpha}{\partial v_\beta^j} = g_{ij}^{\alpha\beta} X^i.$$

Consider now ϕ any solution to the Euler-Lagrange equations (2.14) (which may not be a solution of any ξ). It follows that ϕ satisfies equations (2.20), since ξ is assumed to be an integrable SOPDE.

If we contract equations (2.20) by $X^i \circ \phi^{(1)}$, we obtain

$$(3.7) \quad (X^i \circ \phi^{(1)})(g_{ij}^{\alpha\beta} \circ \phi^{(1)}) \left(\frac{\partial^2 \phi^j}{\partial t^\alpha \partial t^\beta} - \xi_{\alpha\beta}^j \circ \phi^{(1)} \right) = 0.$$

If we replace formula (3.6) in equation (3.7) we obtain

$$0 = \frac{\partial f^\alpha}{\partial v_\beta^j} \circ \phi^{(1)} \left(\frac{\partial^2 \phi^j}{\partial t^\alpha \partial t^\beta} - \xi_{\alpha\beta}^j \circ \phi^{(1)} \right) = \frac{\partial(f^\alpha \circ \phi^{(1)})}{\partial t^\alpha} - \xi_\alpha(f^\alpha) \circ \phi^{(1)},$$

and this formula proves the result. \square

Let us recall the definition of Cartan symmetry for a Lagrangian L , see [47].

Definition 3.5. A vector field $X \in \mathfrak{X}(T_k^1 Q)$ is called a *Cartan symmetry* of the Lagrangian L , if $\mathcal{L}_X \omega_L^\alpha = 0$ for all $\alpha \in \{1, \dots, k\}$ and $\mathcal{L}_X E_L = 0$.

In this case the flow ϕ_t of X transforms solutions to the Euler-Lagrange equations on solutions to the Euler-Lagrange equations, that is, each ϕ_t is a symmetry of the Euler-Lagrange equations, see [47].

From condition $\mathcal{L}_X \omega_L^\alpha = 0$ one obtains that there exists locally defined functions f^α such that $i_X \omega_L^\alpha = df^\alpha$. Thus if X is a Cartan symmetry Lemma 3.4 holds locally.

3.2. Newtonoid vector fields. In this subsection we study some properties of the set of Newtonoid vector fields associated to a SOPDE, generalizing the $k = 1$ case, introduced by Marmo and Mukunda in [36]. Properties of the newtonoid vector fields associated to a SODE and their relations to symmetries and first order variation of geodesics were studied [6]. We extend some of these properties to the case $k > 1$. In Proposition 3.8 we prove that Cartan symmetries of a regular Lagrangian L are Newtonoid vector fields for all corresponding SOPDE $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$.

We fix a SOPDE ξ and consider the following set of vector fields on $T_k^1 Q$

$$(3.8) \quad \mathfrak{X}_\xi = \text{Ker}(J^\alpha \circ \mathcal{L}_{\xi_\alpha}) \subset \mathfrak{X}(T_k^1 Q).$$

The set \mathfrak{X}_ξ can be expressed locally as follows

$$(3.9) \quad \mathfrak{X}_\xi = \left\{ X \in \mathfrak{X}(T_k^1 Q), X = X^i \frac{\partial}{\partial q^i} + \xi_\alpha(X^i) \frac{\partial}{\partial v_\alpha^i} \right\}.$$

Indeed, for a vector field $X \in \mathfrak{X}(T_k^1 Q)$, we have

$$(3.10) \quad \begin{aligned} [\xi_\alpha, X] &= \left[v_\alpha^i \frac{\partial}{\partial q^i} + \xi_{\alpha\beta}^i \frac{\partial}{\partial v_\beta^i}, X^i \frac{\partial}{\partial q^i} + X_\alpha^i \frac{\partial}{\partial v_\alpha^i} \right] \\ &= (\xi_\alpha(X^i) - X_\alpha^i) \frac{\partial}{\partial q^i} + (\xi_\alpha(X_\beta^i) - X(\xi_{\alpha\beta}^i)) \frac{\partial}{\partial v_\beta^i}, \end{aligned}$$

and therefore $J^\alpha \circ \mathcal{L}_{\xi_\alpha}(X) = 0$ if and only if $\xi_\alpha(X^i) = X_\alpha^i$.

Definition 3.6. Consider ξ a SOPDE.

- 1) A vector field $X \in \mathfrak{X}_\xi$ is called a *Newtonoid* for ξ .
- 2) A vector field $X \in \mathfrak{X}(T_k^1 Q)$ is called a *dynamical symmetry* of ξ if $[\xi_\alpha, X] = 0$, for all $\alpha \in \{1, \dots, k\}$.

From formula (3.10) it follows that any dynamical symmetry for a SOPDE ξ is a Newtonoid for ξ . For $k = 1$, the set \mathfrak{X}_ξ was introduced in [36] and it was called the set of Newtonoid vector fields. In the next lemma we provide some properties of the set of Newtonoid vector fields.

Lemma 3.7. For a SOPDE ξ consider the map $\pi_\xi : \mathfrak{X}(T_k^1 Q) \rightarrow \mathfrak{X}(T_k^1 Q)$, given by $\pi_\xi = \text{Id} + J^\alpha \circ \mathcal{L}_{\xi_\alpha}$.

- 1) The map π_ξ satisfies $\pi_\xi \circ \pi_{\xi'} = \pi_\xi$, for any two SOPDEs ξ and ξ' . In particular we have $\pi_\xi^2 = \pi_\xi$ and hence π_ξ is a projector;
- 2) $\text{Im } \pi_\xi = \mathfrak{X}_\xi$, $\text{Ker } \pi_\xi = \mathfrak{X}^v(T_k^1 Q)$, and hence the following sequence is exact

$$0 \rightarrow \mathfrak{X}^v(T_k^1 Q) \xrightarrow{i} \mathfrak{X}(T_k^1 Q) \xrightarrow{\pi_\xi} \mathfrak{X}_\xi \rightarrow 0.$$

- 3) For $f \in C^\infty(T_k^1 Q)$ and $X \in \mathfrak{X}_\xi$, we define the product

$$(3.11) \quad f * X = \pi_\xi(fX) = fX + \xi_\alpha(f)J^\alpha X \in \mathfrak{X}_\xi.$$

The set \mathfrak{X}_ξ is a $C^\infty(T_k^1 Q)$ -module with respect to the $*$ product.

- 4) A vector field X on $T_k^1 Q$ is a Newtonoid for ξ if and only if it has the local expression

$$(3.12) \quad X = X^i(q, v) * \frac{\partial}{\partial q^i}.$$

Proof. Using the definition of the map π_ξ it follows that $\pi_\xi \circ \pi_{\xi'} = \text{Id} + J^\alpha \circ \mathcal{L}_{\xi_\alpha} + J^\alpha \circ \mathcal{L}_{\xi'_\alpha} + J^\alpha \circ \mathcal{L}_{\xi_\alpha} \circ J^\beta \circ \mathcal{L}_{\xi'_\beta}$. Now using the formula $J^\alpha \circ \mathcal{L}_{\xi_\alpha} \circ J^\beta = -J^\beta$ it follows that $\pi_\xi \circ \pi_{\xi'} = \pi_\xi$, which shows that first part of the lemma is true.

Using formula (3.10), we have

$$\pi_\xi \left(X^i \frac{\partial}{\partial q^i} + X_\alpha^i \frac{\partial}{\partial v_\alpha^i} \right) = X^i \frac{\partial}{\partial q^i} + \xi_\alpha(X^i) \frac{\partial}{\partial v_\alpha^i},$$

which shows that $\text{Im } \pi_\xi = \mathfrak{X}_\xi$ and $\text{Ker } \pi_\xi = \mathfrak{X}^v(T_k^1 Q)$.

With the $*$ product defined in formula (3.11), the map π_ξ transfers the $C^\infty(T_k^1 Q)$ -module structure of $(\mathfrak{X}(T_k^1 Q), \cdot)$ to $(\mathfrak{X}_\xi, *)$.

We have that $\text{Im } \pi_\xi = \mathfrak{X}_\xi$. Therefore $X \in \mathfrak{X}_\xi$ if and only if $X = \pi_\xi(X)$. Using the above properties of map π_ξ , a vector field X on $T_k^1 Q$ is locally given as in the last part of formula (3.9) if and only if it is given by formula (3.12). \square

From formula (2.6) it follows that the complete lift $Z^c \in \mathfrak{X}(T_k^1 Q)$ of a vector field $Z \in \mathfrak{X}(Q)$ is a Newtonoid vector field for an arbitrary SOPDE ξ . In the next proposition we will see that the set of Newtonoid vector fields contains also Cartan symmetries.

Proposition 3.8. Consider L a regular Lagrangian on $T_k^1 Q$ and $X \in \mathfrak{X}(T_k^1 Q)$ a Cartan symmetry of L . Then X is a Newtonoid vector field for every $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$.

Proof. Consider $X \in \mathfrak{X}(T_k^1 Q)$ a Cartan symmetry of L and $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$. Since L is regular it follows that ξ is a SOPDE. Moreover, ξ is a solution of the equation $i_{\xi_\alpha} \omega_L^\alpha = dE_L$. If we apply \mathcal{L}_X to both sides of this equation and use the commutation rules we obtain

$$i_{\xi_\alpha} \mathcal{L}_X \omega_L^\alpha - i_{[\xi_\alpha, X]} \omega_L^\alpha = d\mathcal{L}_X E_L.$$

Using now the fact $\mathcal{L}_X \omega_L^\alpha = 0$ and $\mathcal{L}_X E_L = 0$ it follows that

$$(3.13) \quad i_{[\xi_\alpha, X]} \omega_L^\alpha = 0.$$

We will prove now that equation (3.13) implies that $J^\alpha[\xi_\alpha, X] = 0$ and hence X is a Newtonoid vector field for ξ . Using formula (3.10), we have

$$(3.14) \quad [\xi_\alpha, X] = V_\alpha^i \frac{\partial}{\partial q^i} + V_{\alpha\beta}^i \frac{\partial}{\partial v_\beta^i},$$

where $V_\alpha^i = \xi_\alpha(X^i) - X_\alpha^i$ and $V_{\alpha\beta}^i = \xi_\alpha(X_\beta^i) - X(\xi_{\alpha\beta}^i)$. Using formula (2.5), it follows that the k -symplectic 2-forms ω_L^α can be written as follows

$$(3.15) \quad \omega_L^\alpha = a_{ij}^\alpha dq^i \wedge dq^j + g_{ij}^{\alpha\beta} dq^i \wedge dv_\beta^j,$$

where

$$a_{ij}^\alpha = \frac{1}{2} \left(\frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} - \frac{\partial^2 L}{\partial q^i \partial v_\alpha^j} \right), \quad g_{ij}^{\alpha\beta} = \frac{\partial^2 L}{\partial v_\alpha^i \partial v_\beta^j}.$$

If we replace now formulae (3.14) and (3.15) in equation (3.13) we obtain

$$\left(2a_{ij}^\alpha V_\alpha^j - g_{ij}^{\alpha\beta} V_{\alpha\beta}^j \right) dq^i + g_{ij}^{\alpha\beta} V_\alpha^i dv_\beta^j = 0,$$

which implies that $g_{ij}^{\alpha\beta} V_\alpha^i = 0$. Using the fact that the Lagrangian L is regular it follows that $g_{ij}^{\alpha\beta}$ has maximal rank and hence $V_\alpha^i = \xi_\alpha(X^i) - X_\alpha^i = 0$, which shows that X is a Newtonoid vector field for ξ . \square

3.3. Noether's Theorem. For the $k = 1$ case it is well known that Cartan symmetries induce and are induced by constants of motion, and these results are known as Noether's Theorem and its converse. For $k > 1$, Noether's Theorem is also true, each Cartan symmetry induces a conservation law, see Theorem 3.9. However, its converse may not be true. In Proposition 3.11 we discuss when this is the case.

The following theorem is proved in [47, Thm 3.13] using local coordinates. Here we give a direct proof of Noether's Theorem using the Frölicher-Nijenhuis formalism on $T_k^1 Q$. This proof will allow us to discuss also when the converse of Noether's Theorem is true, for $k > 1$. To show that there are cases when the converse of Noether's Theorem is not true, we provide examples of conservation laws that are not induced by Cartan symmetries.

Theorem 3.9. (Noether's Theorem) Consider L a Lagrangian on $T_k^1 Q$ and $X \in \mathfrak{X}(T_k^1 Q)$ a Cartan symmetry for L . Then, there exists (locally defined) functions g^α on $T_k^1 Q$ such that

$$(3.16) \quad \mathcal{L}_X \theta_L^\alpha = dg^\alpha$$

and the following functions

$$(3.17) \quad f^\alpha = \theta_L^\alpha(X) - g^\alpha$$

give a conservation law for the Euler-Lagrange equations.

Proof. Since X is a Cartan symmetry for L it follows that $\mathcal{L}_X \omega_L^\alpha = 0$ and hence the 1-forms $\mathcal{L}_X \theta_L^\alpha$ are closed. Locally, on $T_k^1 Q$, one can find function g^α such that $\mathcal{L}_X \theta_L^\alpha = dg^\alpha$, thus

$$i_X d\theta_L^\alpha + di_X \theta_L^\alpha = dg^\alpha,$$

or equivalently

$$i_X \omega_L^\alpha = d(\theta_L^\alpha(X) - g^\alpha).$$

We will show now that functions $f^\alpha = \theta_L^\alpha(X) - g^\alpha$, in formula (3.17), give a conservation law. We will compute first $\xi_\alpha(f^\alpha)$.

Using formula (3.17) we have

$$(3.18) \quad \begin{aligned} \xi_\alpha(f^\alpha) &= \mathcal{L}_{\xi_\alpha} i_X \theta_L^\alpha - \xi_\alpha(g^\alpha) = i_X \mathcal{L}_{\xi_\alpha} dJ^\alpha L + i_{[\xi_\alpha, X]} dJ^\alpha L - \xi_\alpha(g^\alpha) \\ &= i_X dL + i_{[\xi_\alpha, X]} dJ^\alpha L - \xi_\alpha(g^\alpha). \end{aligned}$$

We apply now i_{ξ_α} to both terms in formula $\mathcal{L}_X dJ^\alpha L = dg^\alpha$, sum over α , and obtain

$$\begin{aligned} \xi_\alpha(g^\alpha) &= i_{\xi_\alpha} dg^\alpha = i_{\xi_\alpha} \mathcal{L}_X dJ^\alpha L = \mathcal{L}_X i_{\xi_\alpha} dJ^\alpha L + i_{[\xi_\alpha, X]} dJ^\alpha L \\ &= \mathcal{L}_X \mathbb{C}(L) + i_{[\xi_\alpha, X]} dJ^\alpha L. \end{aligned}$$

If we replace now, $\xi_\alpha(g^\alpha)$, from the above formula in formula (3.18) we obtain $\xi_\alpha(f^\alpha) = -\mathcal{L}_X(E_L) = 0$. Therefore we have:

$$\xi_\alpha(f^\alpha) = 0, \quad i_X \omega_L^\alpha = df^\alpha.$$

Now using Lemma 3.4 it follows that f^α is a conservation law for L . \square

We have seen that if X is a Cartan symmetry for a Lagrangian L on $T_k^1 Q$ then the functions $f^\alpha \in C^\infty(T_k^1 Q)$, which satisfy the equation $i_X \omega_L^\alpha = df^\alpha$, give a conservation law for L . We say that this conservation law f^α is induced by the Cartan symmetry X . For $k > 1$ there are conservation laws that are not induced by Cartan symmetries. Next we provide such an example.

Example 3.10.

a) We have seen in Example 3.2 that the functions $f^\alpha : T_2^1 \mathbb{R} \rightarrow \mathbb{R}$, given by formula (3.2), give a conservation law for the Euler-Lagrange equations (2.24). We will prove now that this conservation law is not induced by a Cartan symmetry, and hence it will show that the converse of Noether's Theorem 3.9 is not true, unless the assumptions (3.5) are satisfied. Consider $X \in \mathfrak{X}(T_2^1 \mathbb{R})$, locally given by

$$X = Z \frac{\partial}{\partial q} + Z_1 \frac{\partial}{\partial v_1} + Z_2 \frac{\partial}{\partial v_2}$$

Using formulae (2.26), first equation (3.5), for $\alpha = 1$, can be written as follows

$$i_X \omega_L^1 = \sigma(Z dv_1 - Z_1 dq) = df^1 = \frac{\partial f^1}{\partial q} dq + \frac{\partial f^1}{\partial v_1} dv_1 + \frac{\partial f^1}{\partial v_2} dv_2.$$

This implies that $\frac{\partial f^1}{\partial v_2} = 0$, which is not true, since in our case $\frac{\partial f^1}{\partial v_2} = -2\sigma v_1$.

b) Consider the homogeneous isotropic 2-dimensional wave equation

$$(3.19) \quad u_{tt} - cu_{xx} - cu_{yy} = 0.$$

Let us make the following notations $t^1 = t, t^2 = x, t^3 = y$ and $q = u$. The regular Lagrangian function $L \in C^\infty(T_3^1 \mathbb{R})$ for the wave equation (3.19) is

$$(3.20) \quad L = \frac{1}{2} ((v_1)^2 - c(v_2)^2 - c(v_3)^2).$$

Each of the following three sets of functions on $T_3^1 \mathbb{R}$ will give a conservation law for the Lagrangian L in formula (3.20):

$$\begin{aligned} f^1(v_1, v_2, v_3) &= (v_1)^2 + c(v_2)^2 + c(v_3)^2, & f^2(v_1, v_2, v_3) &= -2cv_1v_2, & f^3(v_1, v_2, v_3) &= -2cv_1v_3; \\ f^1(v_1, v_2, v_3) &= 2v_1v_2, & f^2(v_1, v_2, v_3) &= -(v_1)^2 - c(v_2)^2 + c(v_3)^2, & f^3(v_1, v_2, v_3) &= -2cv_2v_3; \\ f^1(v_1, v_2, v_3) &= 2v_1v_3, & f^2(v_1, v_2, v_3) &= -2cv_2v_3, & f^3(v_1, v_2, v_3) &= -(v_1)^2 + c(v_2)^2 - c(v_3)^2. \end{aligned}$$

None of these conservation laws are induced by Cartan symmetries.

Theorem 3.9 shows that any Cartan symmetry of a Lagrangian L induces (locally defined) conservation laws, for $k \geq 1$.

For the case $k = 1$ the converse of this theorem is also true: any conservation law of a Lagrangian is induced by a Cartan symmetry.

In the case $k > 1$ such result is not true anymore, unless we require some extra assumptions. As we have already seen in Example 3.10 there are examples of conservation laws for some Lagrangians, that are not induced by any Cartan symmetries.

Part of the next proposition will show when conservation laws for a Lagrangian are induced by Cartan symmetries.

Proposition 3.11. *Consider $L \in C^\infty(T_k^1 Q)$ a Lagrangian, functions $f^1, \dots, f^k \in C^\infty(T_k^1 Q)$, and a vector field $X \in \mathfrak{X}(T_k^1 Q)$ such that equations (3.5) are satisfied. Then f^α is a conservation law for L if and only if X is Cartan symmetry.*

Proof. In view of Lemma 3.4 we will have to prove that X is a Cartan symmetry if and only if $\xi_\alpha(f^\alpha) = 0$ for all integrable SOPDE $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$.

Using formula (3.5), and the fact that $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$, we have

$$(3.21) \quad X(E_L) = i_X dE_L = -i_X i_{\xi_\alpha} \omega_L^\alpha = i_{\xi_\alpha} i_X \omega_L^\alpha = i_{\xi_\alpha} df^\alpha = \xi_\alpha(f^\alpha).$$

From formula (3.5) it follows that $\mathcal{L}_X \omega_L^\alpha = 0$. Therefore, X is a Cartan symmetry if and only if $X(E_L) = 0$ and, in view of formula (3.21), this is equivalent to $\xi_\alpha(f^\alpha) = 0$. \square

For the case $k = 1$, the regularity condition of the Lagrangian implies that the Poincaré-Cartan 2-form ω is a symplectic form and hence equation (3.5) always has a unique solution. In the case $k > 1$, for some given functions $f^\alpha \in C^\infty(T_k^1 Q)$, the system (3.5) is overdetermined and it may not have solutions $X \in \mathfrak{X}(T_k^1 Q)$. We will provide examples when this is the case.

Example 3.12. 1) Let us consider the following Lagrangians $L : T_2^1 \mathbb{R} \rightarrow \mathbb{R}$:

$$(a) \quad L(q, v_1, v_2) = \frac{1}{2}(\sigma v_1^2 - \tau v_2^2), \quad (b) \quad L(q, v_1, v_2) = \sqrt{1 + (v_1)^2 + (v_2)^2}.$$

The vector field $X = \partial/\partial q$ on $T_k^1 Q$ is a Cartan symmetry for both Lagrangians and the corresponding conservation laws are

$$(a) \quad f^1 = \sigma v_1, \quad f^2 = -\tau v_2, \quad (b) \quad f^1 = \frac{v_1}{\sqrt{1 + (v_1)^2 + (v_2)^2}}, \quad f^2 = \frac{v_2}{\sqrt{1 + (v_1)^2 + (v_2)^2}}.$$

The above Lagrangians correspond to the vibrating string equations and the equation of minimal surfaces, respectively, see [40, 45].

2) For the Lagrangian $L : T_3^1 \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$L(q, v_1, v_2, v_3) = \frac{1}{2}((v_1)^2 + (v_2)^2 + (v_3)^2),$$

the vector field $X = \partial/\partial q$ is a Cartan symmetry, and the induced conservation law is

$$f^1(v_1) = v_1, \quad f^2(v_2) = v_2, \quad f^3(v_3) = v_3.$$

The Euler-Lagrange equations corresponding to L are the Laplace equations.

3) For the Lagrangian $L : T_2^1 \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$L(q^1, q^2, v_1^1, v_1^2, v_2^1, v_2^2) = \left(\frac{1}{2} \lambda + \nu \right) [(v_1^1)^2 + (v_2^2)^2] + \frac{1}{2} \nu [(v_2^1)^2 + (v_1^2)^2] + (\lambda + \nu) v_1^1 v_2^2,$$

the vector field $X = \partial/\partial q^1 + \partial/\partial q^2$ is again a Cartan symmetry. The induced conservation law is

$$f^1 = (\lambda + 2\nu) v_1^1 + \nu v_1^2 + (\lambda + \nu) v_2^2, \quad f^2 = (\lambda + \nu) v_1^1 + \nu v_2^1 + (\lambda + 2\nu) v_2^2.$$

The Euler-Lagrange equations corresponding to L are the Navier equations, see [40, 45]

In Proposition 3.8 we have seen that Cartan symmetries are Newtonoid vector fields. Next theorem shows that under some assumptions Newtonoid vector fields provide Cartan symmetries and hence conservation laws. This theorem generalizes the result obtained in the case $k = 1$ by Marmo and Mukunda [36] for regular Lagrangians.

Theorem 3.13. *Consider L a regular Lagrangian on $T_k^1 Q$. We assume that there exists $X \in \mathfrak{X}(T_k^1 Q)$ and $g^\alpha \in C^\infty(T_k^1 Q)$ such that*

$$(3.22) \quad \pi_\xi(X)(L) = \xi_\alpha(g^\alpha), \forall \text{ SOPDE } \xi_\alpha.$$

Then, it follows:

- 1) *If $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$ we have that $\pi_\xi(X)$ is a Cartan symmetry for L .*
- 2) *The functions $f^\alpha = \theta_L^\alpha(X) - g^\alpha$ give a conservation law for L .*

Proof. 1) We have to prove that the following two conditions are satisfied

$$(a) \quad \mathcal{L}_{\pi_\xi(X)}\omega_L^\alpha = 0, \quad (b) \quad \mathcal{L}_{\pi_\xi(X)}E_L = 0.$$

(a) For each $\alpha \in \{1, \dots, k\}$ we denote the 1-forms

$$(3.23) \quad \eta^\alpha = \mathcal{L}_{\pi_\xi(X)}\theta_L^\alpha - dg^\alpha.$$

First we show that

$$i_{V_\alpha}\eta^\alpha = 0, \quad L_{\xi_\alpha}\eta^\alpha(V) = 0$$

for arbitrary vertical vector fields V, V_1, V_2, \dots, V_k . First condition above is equivalent to the fact that $\eta^\alpha = \eta_i^\alpha dq^i$ are semi-basic 1-forms. Moreover, using the fact that $\mathcal{L}_{\xi_\alpha}\eta^\alpha = \xi_\alpha(\eta_i^\alpha) dq^i + \eta_i^\alpha dv_\alpha^i$, it follows that the second condition will imply $\eta^\alpha = 0$.

Let $\xi_\alpha \in \mathfrak{X}(T_k^1 Q)$ be a SOPDE and V_α be vertical vector fields. It follows that $\xi'_\alpha = \xi_\alpha - V_\alpha$ are also SOPDES. Using the fact that θ_L^α are semi-basic 1-forms and V_α are vertical vector fields, we have that $i_{V_\alpha}\theta_L^\alpha = 0$. Therefore, making use of the corresponding commutation rules, we have

$$\begin{aligned} i_{V_\alpha}\eta^\alpha &= i_{V_\alpha}\mathcal{L}_{\pi_\xi(X)}\theta_L^\alpha - i_{V_\alpha}dg^\alpha = i_{V_\alpha}\mathcal{L}_{\pi_\xi(X)}\theta_L^\alpha - \mathcal{L}_{\pi_\xi(X)}i_{V_\alpha}\theta_L^\alpha - i_{V_\alpha}dg^\alpha \\ &= i_{[V_\alpha, \pi_\xi(X)]}\theta_L^\alpha - i_{V_\alpha}dg^\alpha = i_{J^\alpha[V_\alpha, \pi_\xi(X)]}dL - i_{V_\alpha}dg^\alpha \\ &= i_{\pi_\xi(X)}dL - i_{\pi_{\xi'}(X)}dL - i_{V_\alpha}dg^\alpha = \xi_\alpha(g^\alpha) - \xi'_\alpha(g^\alpha) - V_\alpha(g^\alpha) = 0. \end{aligned}$$

In the above calculations we did use the fact that $\pi_\xi(X) - \pi_{\xi'}(X) = J^\alpha[V_\alpha, \pi_\xi(X)]$ and the fact that the SOPDES ξ and ξ' satisfy the hypothesis (3.22).

We fix now $\xi \in \mathfrak{X}_L^k(T_k^1 Q)$, and since L is regular this means that $\mathcal{L}_{\xi_\alpha}\theta_L^\alpha = dL$. Using the notation (3.23) we have

$$\begin{aligned} \mathcal{L}_{\xi_\alpha}\eta^\alpha &= \mathcal{L}_{\xi_\alpha}\mathcal{L}_{\pi_\xi(X)}\theta_L^\alpha - \mathcal{L}_{\xi_\alpha}dg^\alpha = \mathcal{L}_{[\xi_\alpha, \pi_\xi(X)]}\theta_L^\alpha + \mathcal{L}_{\pi_\xi(X)}\mathcal{L}_{\xi_\alpha}\theta_L^\alpha - \mathcal{L}_{\xi_\alpha}dg^\alpha \\ &= i_{[\xi_\alpha, \pi_\xi(X)]}\omega_L^\alpha + \mathcal{L}_{\pi_\xi(X)}dL - d\xi_\alpha(g^\alpha) = i_{[\xi_\alpha, \pi_\xi(X)]}\omega_L^\alpha. \end{aligned}$$

Using the fact that $[\xi_\alpha, \pi_\xi(X)]$ is a vertical vector fields, and the k -symplectic structure in formula (2.5) vanishes on pairs of vertical vector fields, it follows that for an arbitrary vertical vector field V we have

$$\mathcal{L}_{\xi_\alpha}\eta^\alpha(V) = \omega_L^\alpha([\xi_\alpha, \pi_\xi(X)], V) = 0.$$

Hence, we proved that $\eta^\alpha = 0$, which means that

$$(3.24) \quad \mathcal{L}_{\pi_\xi(X)}\theta_L^\alpha = dg^\alpha.$$

If we take the exterior derivative in the above formula it follows that $\mathcal{L}_{\pi_\xi(X)}\omega_L^\alpha = 0$.

(b) In order to prove that $\pi_\xi(X)$ is a Cartan symmetry it remains to show that $\pi_\xi(X)(E_L) = 0$. For this we use the fact that $\mathbb{C} = J^\alpha(\xi_\alpha)$ and hence $\mathbb{C}(L) = i_{\mathbb{C}}dL = i_{\xi_\alpha}\theta_L^\alpha$. Therefore,

$$\begin{aligned} \pi_\xi(X)(E_L) &= \pi_\xi(X)(\mathbb{C}(L)) - \pi_\xi(X)(L) = \mathcal{L}_{\pi_\xi(X)}i_{\xi_\alpha}\theta_L^\alpha - \mathcal{L}_{\pi_\xi(X)}L \\ &= i_{\xi_\alpha}(\mathcal{L}_{\pi_\xi(X)}\theta_L^\alpha - dg^\alpha) = 0. \end{aligned}$$

2) So far we have proved that $\pi_\xi(X)$ is a Cartan symmetry and it satisfies formula (3.24). Using the fact $J^\alpha \circ \pi_\xi = J^\alpha$ and Noether's theorem 3.9, it follows that the functions $f^\alpha = \theta_L^\alpha(\pi_\xi(X)) - g^\alpha = \theta_L^\alpha(X) - g^\alpha$ give a conservation law for L . \square

Theorem 3.13 extends the results in Corollary 3.15 from [47]. Indeed if $X = Z^C$ for some $Z \in \mathfrak{X}(M)$ and $g^\alpha \in C^\infty(M)$ the condition (3.22) becomes $Z^C(L) = v_\alpha^i \partial g^\alpha / \partial q^i$. It follows that Z^C is a Cartan symmetry and the functions $f^\alpha = Z^{v_\alpha}(L) - g^\alpha$ define a conservation law.

Acknowledgments. The work of IB was supported by the Romanian National Authority for Scientific Research, CNCS UEFISCDI, project number PN-II-ID-PCE-2012-4-0131.

We acknowledge the financial support of the Ministerio de Economía y Competitividad (Spain), projects MTM2011-22585 and MTM2011-15725-E.

We express our thanks to the referees for their comments and suggestions.

REFERENCES

- [1] R.A. Abraham, J.E. Marsden. Foundations of Mechanics, (Second Edition), Benjamin-Cummings Publishing Company, New York, (1978).
- [2] V.I. Arnold. Mathematical methods of classical mechanics, *Graduate Texts in Mathematics* **60**. Springer-Verlag, New York-Heidelberg, (1978).
- [3] A. Awane. k -symplectic structures, *J. Math. Phys.* **33** (1992), 4046-4052.
- [4] A. Awane, M. Goze. Pfaffian systems, k -symplectic systems, Kluwer Academic Publishers, Dordrecht (2000).
- [5] E. Binz, J. Sniatycki, H. Fischer. Geometry of classical fields, North-Holland Mathematics Studies, 154(1988).
- [6] I. Bucataru, O.A. Constantinescu, M.F. Dahl. A geometric setting for systems of ordinary differential equations, *International Journal of Geometric Methods in Modern Physics*, **8** (6) (2011), 1292–1327.
- [7] I. Bucataru, M.F. Dahl. A complete lift for semisprays, *International Journal of Geometric Methods in Modern Physics*, **7**(2) (2010), 267–287.
- [8] F. Cantrijn, A. Ibort, M. de León. On the geometry of multisymplectic manifolds, *J. Austral. Math. Soc. Ser. A* **66** (1999), 303-330.
- [9] F. Cantrijn, A. Ibort, M. de León. Hamiltonian structures on multisymplectic manifolds, *Rend. Sem. Mat. Univ. Politec. Torino*, **54** (1996), 225-236.
- [10] M. Crampin, F.A.E. Pirani. Applicable differential geometry. Cambridge University Press, 1986.
- [11] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy. Multivector Field Formulation of Hamiltonian Field Theories: Equations and Symmetries, *J. Phys. A: Math. Gen.* **32**(48) (1999) 8461-8484.
- [12] P.L. García, A. Pérez-Rendón. Symplectic approach to the theory of quantized fields, I, *Comm. Math. Phys.* **13** (1969) 24-44.
- [13] P.L. García, A. Pérez-Rendón. Symplectic approach to the theory of quantized fields, II, *Arch. Ratio. Mech. Anal.* **43** (1971), 101-124.
- [14] G. Giachetta, L. Mangiarotti, G. Sardanashvily. New Lagrangian and Hamiltonian Methods in Field Theory, *World Scientific Pub. Co* , Singapore (1997).
- [15] H. Goldschmidt, S. Sternberg. The Hamilton-Cartan formalism in the calculus of variations, *Ann. Inst. Fourier* **23** (1973), 203-267.
- [16] M.J. Gotay. An exterior differential systems approach to the Cartan form, *Symplectic geometry and mathematical physics. (Aix-en-Provence, 1990)*. Progr. Math., 99, Birkhäuser Boston, Boston, MA, 1991, pp. 160-188.
- [17] M.J. Gotay. A multisymplectic framework for classical field theory and the calculus of variations, I. Covariant Hamiltonian formalism, *Mechanics, analysis and geometry: 200 years after Lagrange*. North-Holland Delta Ser., North-Holland, Amsterdam, 1991, pp. 203-235.
- [18] M.J. Gotay. A multisymplectic framework for classical field theory and the calculus of variations, II. Space + time decomposition, *Differential Geom. App.* **1** (1991), 375-390.
- [19] M. J. Gotay, J. Isenberg, J. E. Marsden. Momentum Maps and Classical Relativistic Fields, Part I: Covariant Field Theory, arXiv:physics/9801019v2 (2004). Part II: Canonical analysis of Field Theories, arXiv:math-ph/0411032v1 (2004).
- [20] C. Günther. The polysymplectic Hamiltonian formalism in field theory and calculus of variations I: The local case, *J. Differential Geom.* **25** (1987) 23-53.
- [21] I. V. Kanatchikov. Canonical structure of classical field theory in the polymomentum phase space, *Rep. Math. Phys.* **41**(1) (1998) 49–90.
- [22] J. Kijowski. A finite-dimensional canonical formalism in the classical field theory, *Comm. Math. Phys.* **30** (1973), 99-128.
- [23] J. Kijowski, W. Szczyrba. Multisymplectic manifolds and the geometrical construction of the Poisson brackets in the classical field theory, *Géométrie symplectique et physique mathématique* (Colloq. International C.N.R.S., Aix-en-Provence, 1974) (1974) 347-349.
- [24] J. Kijowski, W. M. Tulczyjew. A symplectic framework for field theories. *Lecture Notes in Physics*, **107**. Springer-Verlag, New York, 1979.
- [25] I. Kolár, P.W. Michor, J. Slovák. Natural operations in differential geometry, *Springer-Verlag*, 1993.

- [26] D.D. Kosambi. Systems of partial differential equations of the second order, *Quart. J. Math.*, **19**(1948), 204–219.
- [27] M. de León, D. Martín de Diego, M. Salgado, S. Vilariño. Nonholonomic constraints in k-symplectic Classical Field Theories. *International Journal of Geometric Methods in Modern Physics*, **5**(5) (2008) 799–830.
- [28] M. de León, D. Martín de Diego, M. Salgado, S. Vilariño. k-symplectic formalism on Lie algebroids. *J. Phys. A: Math. Theor.* **42** (2009) 385209 (31 pp)
- [29] M. de León, D. Martín de Diego. Symmetries and Constant of the Motion for Singular Lagrangian Systems, *Int. J. Theor. Phys.* **35**(5) (1996) 975–1011.
- [30] M. de León, D. Martín de Diego, A. Santamaría-Merino. Symmetries in classical field theories, *Int. J. Geom. Meth. Mod. Phys.* **1**(5) (2004) 651–710.
- [31] M. de León, I. Méndez, M. Salgado. p -almost tangent structures, *Rend. Circ. Mat. Palermo, Serie II XXXVII* (1988), 282–294.
- [32] M. de León, I. Méndez, M. Salgado. Integrable p -almost tangent structures and tangent bundles of p^1 -velocities, *Acta Math. Hungar.* **58**(1-2) (1991), 45–54.
- [33] M. de León, E. Merino, J.A. Oubiña, P. Rodrigues, M. Salgado. Hamiltonian systems on k -cosymplectic manifolds, *J. Math. Phys.* **39**(2) (1998) 876–893.
- [34] M. de León, E. Merino, M. Salgado. k -cosymplectic manifolds and Lagrangian field theories, *J. Math. Phys.* **42**(5) (2001) 2092–2104.
- [35] H. Marañón. Simetries d’equacions diferencials. Aplicació als sistemes k-simplèctics, Treball Fi de Master Matematica Aplicada. 2008 Department of Applied Mathematics IV. Technical University of Catalonia (UPC).
- [36] G. Marmo, N. Mukunda. Symmetries and constants of the motion in the Lagrangian formalism on TQ: beyond point transformations, *Nuovo Cim. B*, **92** (1986) 1–12.
- [37] J.C. Marrero, N. Román-Roy, M. Salgado, S. Vilarino. On a kind of Noether symmetries and conservation laws in k-cosymplectic Field Theory, *Journal of Mathematical Physics* **52**, 022901 (2011), 20 pp
- [38] A. Morimoto. Liftings of some types of tensor fields and connections to tangent p^r -velocities, *Nagoya Qath. J.* **40** (1970) 13–31.
- [39] F. Munteanu, A. M. Rey, M. Salgado. The Günther’s formalism in classical field theory: momentum map and reduction, *J. Math. Phys.* **45**(5) (2004) 1730–1751.
- [40] M. C. Muñoz-Lecanda, M. Salgado, S. Vilariño. k-symplectic and k-cosymplectic Lagrangian field theories: some interesting examples and applications. *International Journal of Geometric Methods in Modern Physics*, **7**(4) (2010) 669–692.
- [41] L.K. Norris. Generalized symplectic geometry on the frame bundle of a manifold, *Proc. Symp. Pure Math.* **54**, Part 2 (Amer. Math. Soc., Providence RI, 1993), 435–465.
- [42] L.K. Norris. Symplectic geometry on T^*M derived from n -symplectic geometry on LM . *J. Geom. Phys.* **13** (1994) 51–78.
- [43] L.K. Norris. Schouten-Nijenhuis Brackets, *J. Math. Phys.* **38** (1997) 2694–2709.
- [44] L. K. Norris. n -symplectic algebra of observables in covariant Lagrangian field theory, *J. Math. Phys.* **42**(10) (2001) 4827–4845.
- [45] P.J. Olver. Applications of Lie groups to differential equations, *Graduate Texts in Mathematics*, **107**. Springer-Verlag, New York, 1986.
- [46] N. Román-Roy, A. M. Rey, M. Salgado, S. Vilariño. On the k-Symplectic, k-Cosymplectic and Multisymplectic Formalism of Classical Field Theories, *Journal of Geometric Mechanics* **3**(1), March 2011
- [47] N. Román-Roy, M. Salgado, S. Vilariño. Symmetries and Conservation Laws in Günter k-symplectic formalism of Field Theory, *Reviews in Mathematical Physics*, **19** (10) (2007), 1117–1147.
- [48] J. Sniatycki. On the geometric structure of classical field theory in Lagrangian formulation, *Math. Proc. Cambridge Philos. Soc.* **68** (1970) 475–484.
- [49] W.M. Tulczyjew. Hamiltonian systems, Lagrangian systems and the Legendre transformation, *Symposia Mathematica* **16** (1974) 247–258.
- [50] K.Yano, S. Ishihara. Tangent and cotangent bundles, *Marcel Dekker, Inc.*, 1973.

LUCÍA BUA, DEPARTAMENTO DE XEOMETRÍA E TOPOLOXÍA, FACULTADE DE MATEMÁTICAS, UNIVERSIDADE DE SANTIAGO DE COMPOSTELA, SANTIAGO DE COMPOSTELA 15782, SPAIN

IOAN BUCATARU, FACULTY OF MATHEMATICS, UNIVERSITATEA "ALEXANDRU IOAN CUZA" DIN IAȘI, IAȘI, 700506, ROMANIA

URL: <http://www.math.uaic.ro/~bucataru/>

MODESTO SALGADO, DEPARTAMENTO DE XEOMETRÍA E TOPOLOXÍA, FACULTADE DE MATEMÁTICAS, UNIVERSIDADE DE SANTIAGO DE COMPOSTELA, SANTIAGO DE COMPOSTELA 15782, SPAIN

URL: <http://xtsunxet.usc.es/modesto/>